Fermionic Light in Common Optical Media

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Recent experiments have proved that the response to short laser pulses of common optical media, such as air or oxygen, can be described by focusing Kerr and higher order nonlinearities of alternating signs. Such media support the propagation of steady solitary waves. We argue both numerically and analytical computations that the low-power fundamental bright solitons satisfy an equation of state which is similar to that of a degenerate gas of fermions at zero temperature. Considering, in particular, the propagation in both O$_2$ and air, we also find that the high-power solutions behave like droplets of ordinary liquids. We then show how a grid of the fermionic light bubbles can be generated and forced to merge in a liquid droplet. This leads us to propose a set of experiments aimed at the production of both the fermionic and liquid phases of light, and at the demonstration of the transition from the former to the latter.

In suitable optical media, light has been argued to acquire material properties. A long-known example is the equivalence of the paraxial propagation of a laser pulse in a Kerr medium with the time evolution of a superfluid Bose-Einstein condensate, due to the identity of the nonlinear Schrödinger equation with the Gross-Pitaevskii equation [1]. More recently, optical induction has been used to create photonic crystals [2], a photonic system has been designed that may undergo a Mott insulator to superfluid quantum phase transition [3], and soliton solutions for light propagation in cubic-quintic (CQ) nonlinear media have been shown to behave like ordinary liquids [4,5].

On the other hand, recent experimental and theoretical works have proved that the response to ultrashort laser pulses of common optical media, such as air or oxygen, can be described by focusing Kerr [6] and higher order nonlinearities of alternating signs [7], which have also been argued to provide the main mechanism in filament stabilization, instead of the plasma defocusing [7].

In this Letter, we demonstrate by analytical and numerical computations that such media can support the propagation of steady solitary waves that appear in two clearly different phases. The low-power solitons are governed by the same equation of state as a degenerate gas of fermions. We will call such a system “fermionic light.” On the other hand, the high-power localized states satisfy the Young-Laplace (YL) equation that governs the formation of droplets in ordinary liquids, similarly to the result that was recently obtained for the CQ model [5]. We also show how to generate a grid of fermionic light bubbles and make it turn into a liquid light droplet.

We will consider the (paraxial) propagation along the $z$ direction of a linearly polarized laser beam, so that the complex electric field component $\Psi(x, y, z)$ satisfies the nonlinear Schrödinger equation

\[
\frac{i}{\partial z} \Psi + \frac{1}{2k_0 n_0} \nabla_\perp^2 \Psi + k_0 \Delta n \Psi = 0, \tag{1}
\]

where $n_0$ is the linear refractive index of the medium, $\nabla_\perp = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the transverse Laplace operator, and $k_0 = 2\pi / \lambda_0$ is the mean wave number in vacuum, where $\lambda_0$ is the central wavelength of the laser source which will be fixed to $\lambda_0 = 800$ nm throughout this Letter as in the experiment of Ref. [7]. For the optical media that have been studied in Ref. [7], the nonlinear correction $\Delta n$ to the refractive index can be expanded as

\[
\Delta n \approx n_2 |\Psi|^2 + n_4 |\Psi|^4 + n_6 |\Psi|^6 + n_8 |\Psi|^8, \tag{2}
\]

with alternating sign coefficients $n_2, n_6 > 0$ and $n_4, n_8 < 0$ that contribute to focusing and defocusing, respectively. Taking into account that the values of the second-order dispersion and multiphoton-absorption coefficients for air are $\kappa \approx 0.2$ fs$^2$/cm and $\beta \approx 1.27 \times 10^{-12}$ cm$^3$/W$^9$, respectively, we have checked that both effects do not lead to significant corrections in our results. To be concrete, we will perform most of the numerical calculations in the case of O$_2$ as the propagation medium, taking the mean values obtained in the experiment [7], $n_2 = 1.6 \times 10^{-19}$ cm$^2$/W, $n_4 = -5.2 \times 10^{-33}$ cm$^4$/W$^2$, $n_6 = 4.8 \times 10^{-46}$ cm$^6$/W$^6$, $n_8 = -2.1 \times 10^{-59}$ cm$^8$/W$^4$. For comparison, however, we will also mention the results that we obtain for air, using the corresponding values for the coefficients $n_{2\alpha}$ that are also given in Ref. [7].

We will search for finite localized solutions of Eq. (1) of the form $\Psi(x, y, z) = \Phi(r)e^{-i\mu z}$, where $r = \sqrt{x^2 + y^2}$ and $\mu$ is the propagation constant. Figure 1 shows the result of our numerical computation for $\Phi(0)$ in the existence domain of such solitons in oxygen ($\mu_{\infty} < \mu < 0$), where $\mu_{\infty} = -0.096$ cm$^{-1}$, and for the radial profiles $\Phi(r)$ of three of them (see left-hand inset in Fig. 1), corresponding to the low-power (black solid line), moderate-power...
For our optical system, contributions proportional method with the ansatz (black solid line), corresponding to small values of pressure regions, because this is a necessary condition for the radial distributions display both positive and negative values of $A_j$. We express the transverse spatial potential of an equivalent thermodynamical two-dimensional system of a degenerate Fermi gas in the central region around $R=0$, which is arbitrarily large in the limit $\mu \rightarrow 0$ (corresponding to large radius $R \rightarrow \infty$). For these reasons, we will call “fermionic” the phase where the pressure is proportional to $p^2$.

Note that in the limit $\mu \rightarrow 0$ our variational computation gives a constant $N = \frac{2\pi}{k_B n_0 R}$, which is consistent with the known result for the power flow leading to the collapse threshold in a Kerr medium [10]. The magnitude of this power $N$ lies in the range of few GW in both $O_2$ and air, and can be interpreted as the threshold for the existence of the fermionic light solitons.

Figure 2 shows the numerical computation of $p_c$ as a function of either $R$ (black solid line) or $\rho_c$ (inset, black solid line) for all the nodeless solitary states of the model. The lower branch, corresponding to the low-power solitons, is in excellent agreement with the dependence described by Eq. (4), as it can be inferred from the fitting (red dotted) straight line with slope $s_{\text{low}} = -4$. Furthermore, in the inset of Fig. 2 we also show the qualitative agreement between theory and numerics by comparing $p_c$ with $\rho_c$ instead of $R$. In this case, the slope of the straight line is $s_{\text{inset}} = 2$, thus demonstrating the quadratic dependence on $\rho_c$ given by Eq. (4). However, the correct numerical values of the constants are $a = 2.5$ and $b = 0.29$, in reasonable agreement with the result of the variational method given above. We have checked numerically that the asymptotic behavior represented by the red lines in Fig. 2 (dotted line and dashed line in the inset) is practically independent of the higher order nonlinearities $n_{2q}$ ($q > 1$), as predicted by the theory. In particular, these results can be directly applied to both Kerr and CQ models.

On the first hand, the high-power localized solutions exhibit top-flat profiles with an inhomogeneous negative pressure profile on the border, similar to those of the solitons appearing in the CQ model [5]. As a

$$p_c = \frac{a}{k_B n_0 n_2} \frac{1}{R^4} = b k_B n_0 \rho_c^2,$$\hspace{1cm}(4)$$

with $a = 4$ and $b = 1/4$. This relation is similar to the equation of state of a degenerate gas of fermions of mass $m$ at zero temperature. In fact, if we apply the general definition of the Fermi momentum $p_F$ to a two-dimensional system, we obtain $p_F = h\sqrt{2\pi \rho}$, with $\rho$ the density of the Fermi gas. As a consequence, the pressure, defined as the average force on a unit orthogonal line in the gas, can be obtained from the average kinetic energy as follows:

$$p = \rho \langle E_{\text{kin}} \rangle = \frac{\rho}{2m} \int_0^{p_F} p^2 P dp = \frac{\pi h^2}{2m} \rho^2,$$\hspace{1cm}(5)$$

which shows the same dependence with $p^2$ as Eq. (4). This proves the formal analogy of our low-power solitons with a degenerate Fermi gas in the central region around $R = 0$, which is arbitrarily large in the limit $\mu \rightarrow 0$ (corresponding to large radius $R \rightarrow \infty$). For these reasons, we will call “fermionic” the phase where the pressure is proportional to $p^2$.

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to a higher intensity, propagation in air, the liquid light phase would correspond

the YL equation (green dashed line) with very good accuracy

state. In fact, we have checked that qualitatively similar

separated phases, satisfying two different equations of

light beams in media like oxygen can occur in two clearly

and the YL equation would still be valid.

As we have seen above, the propagation of self-guided

light beams in media like oxygen can occur in two clearly

separated phases, satisfying two different equations of

state. In fact, we have checked that qualitatively similar

results can also be obtained for the CQ case, and occur

whenever the nonlinear refractive index displays a single

well-defined maximum as a function of the intensity. It

would then be interesting to demonstrate the possibility

to a transition between the fermionic bubbles and the

liquid droplets of light. A suggestive analogy is that of

the collapse of a star, that occurs when the gravitational

interaction overcomes the Fermi pressure of the electrons.

We can obtain a qualitatively similar result in the case of

light propagation in oxygen, by compressing the fermionic

bubbles using a harmonic potential, leading to the genera-

tion of a liquid light droplet. Figure 3 shows the result of

our simulation in O\textsubscript{2}. The initial state (see left-hand snap-

shot in Fig. 3) consists of a regular grid of fermionic light

bubbles with $\mu/\mu_\infty = 0.3$ (see their radial profile in

Fig. 1), with a separation between nearest neighbors

$\Delta_\xi, \chi = 40$. We include an external harmonic potential

$V(\xi, \chi) = \frac{K}{2}(\xi^2 + \chi^2)$ with $K = 5 \times 10^{-5}$ in Eq. (1)

in order to induce a net force acting on the grid with the

aim of making all the fermionic solitons collide in the

center of the computational window $(\xi, \chi) = 0$.

This parabolic potential can be obtained by inducing in the

medium a “gas lens”[12], which can be constructed

with an electrically heated pipe through which passes a

laminar flow of gas. By controlling the differential heating

at the boundaries and the velocity of the flow, a parabolic

FIG. 2 (color online). Plot of the logarithms of $p_c$ vs $R$ for the

fundamental solitons (black line). The fermionic behavior of Eq.

(4) (red dotted line) and the liquid YL equation (green dashed line) are compared with the numerics. The labeled points corre-

spond to the same eigenstates displayed in Fig. 1. For each value

of $R$, two different outcomes are possible depending on the

power, corresponding to two different phases of light.

Inset: Plot of $\ln(p_c)$ vs $\ln(R)$ (black solid line) overlapped with Eq. (4) (red dashed line) for comparison.

consequence, the gradient term in Eq. (1) can be neglected

close to the origin, and we get $\mu = -k_0 \Delta n(0) =

-k_0 \sum_{q=1}^{4} n_{2q} \Phi(0)^{2q}$. By generalizing the argument

of Ref. [5] including the higher order nonlinearities, we ob-

tain that these states obey the celebrated YL equation [11],

$p_c = 2 \sigma/R$, describing the behavior of usual liquid dro-

plets. The value of the surface tension is

$$
\sigma = \frac{n_2}{\sqrt{2 n_0 n_4}} \int_0^{\Phi_\infty} \left( -\mu_\infty \Phi^2 - k_0 \sum_{q=1}^{4} \frac{n_{2q} \Phi^{2q+1}}{q+1} \right)^{1/2} d\Phi,
$$

(6)

where $\mu_\infty$ and $\Phi_\infty$ are the asymptotic values corresponding

to the $R \to \infty$ droplet, which can be computed by solving the

equation $p_c = 0$ (neglecting the Laplacian term).

For the propagation in oxygen, we have obtained the fol-

lowing variational estimations: $\mu_\infty = -0.247(k_0 n_2 n_4^{-1}) =

-0.096 \text{ cm}^{-1}$, $\Phi_\infty = 0.712(n_2 n_4^{-1}) = 2.19 \times 10^{13} \text{ W/cm}^2$,

$\sigma = 0.0715(n_2 n_4^{-3/2} n_0^{-1/2}) = 4.88 \times 10^9 \text{ W/cm}^2$. These re-

sults, obtained assuming a top-flat function, are in excellent

agreement with the computation given in Fig. 1. In Fig. 2, we

show that our numerical solution in the case of O\textsubscript{2} satisfies

the YL equation (green dashed line) with very good accuracy

for a wide range of values of $\mu$. On the other hand, for the

propagation in air, the liquid light phase would correspond

to a higher intensity, $[\Phi_\infty^2]_0 = 2.98 \times 10^{13} \text{ W/cm}^2$, with

$[\mu_\infty]_0 = -0.116 \text{ cm}^{-1}$ and $[\sigma]_0 = 7.16 \times 10^9 \text{ W/cm}^2$, and the YL equation would still be valid.

As we have seen above, the propagation of self-guided

light beams in media like oxygen can occur in two clearly

separated phases, satisfying two different equations of

state. In fact, we have checked that qualitatively similar

FIG. 3 (color online). Isosurface intensity plot of the dynamical

phase transition from a square grid of fermionic light solitons
to an individual liquid light soliton in O\textsubscript{2}. All solitons in the grid

are eigenstates with $\mu/\mu_\infty = 0.3$ displayed in Fig. 1. They are

forced to merge in the center of the computational window by

means of an external harmonic potential. We can see three

pseudocolor plots displaying the initial condition (left, $\eta = 0$),

the collapse of the light bubbles (middle, $\eta = 200$), and the final

state (right, $\eta = 500$). In the latter we observe a flattop soliton

with radius $R \approx 10$ and $p_c \approx 0.76$ arising after the massive

coalesscence of the fermionic solitons. The width of the square

spatial domain displayed is $w_{\xi, \chi} = 200$. 

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that the demonstration of the existence of both fermionic and liquid light and the phase transition between them would be an affordable challenge in real experiments.

In conclusion, we have proved that common media (O\(_2\), air) can support the propagation of solitary waves that appear in two clearly different phases with unequal physical properties, namely, the low-power “fermionic” light, satisfying an equation of state similar to that of a degenerate gas of fermions, and the high-power “liquid” light, obeying the YL equation. We have then shown how a grid of the fermionic light bubbles can be generated and forced to merge in a liquid droplet. We think that the possible experimental validation of our proposal could also provide an independent way to corroborate the deep change in the understanding of the filamentation process in gases that was proposed in Ref. [7]. Furthermore, these results in air pave the way for the improvement of recent experiments on laser-induced water condensation [14], built on top of these new robust light distributions.

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